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ON DIMENSIONAL INVARIANCE

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I. Introduction

A law, or indeed any mathematical relation between measurable variables, is called dimensionally invariant when its form is invariant to changes in (coherent) units of measurement. All the known fundamental laws of physics and economics are dimensionally invariant. For example

$$E = mc^2 \tag{1}$$

where E is energy, m is mass, and c is the velocity of light, is true no matter what coherent units are chosen for the dimensions of length, mass and time; and Walras' Law

$$\Sigma_1^n p_i q_i = 0, \tag{2}$$

where q_i is the market excess demand for the ith commodity (including money and other financial assets) and p_i its price, is true for all units of money and the n commodities. On the other hand, an expression like

$$\Sigma_1^n p_i q_i (1 - q_i) = 0 \tag{3}$$

is not dimensionally invariant. This expression holds in general economic equilibrium, where each $q_i \leqslant$ o and where $q_i < 0$ implies $p_i = 0$ (i.e. in equilibrium there can be excess supplies (negative excess demands) but only of free goods, and no positive excess demands), and by a suitable choice of units it can be made to hold in a particular disequilibrium as well. Thus eqn. (3) might accurately describe the phe-

* The views here presented are those of the author, and are not necessarily the official views of the Federal Reserve Bank of Dallas or any other part of the Federal Reserve System.

nomena when (say) wheat is measured in tons, but not when it is measured in bushels. For an accurate description in the latter case we should have to change the form of the expression in an unknown manner. To those who use dimensional analysis and other metrological principles in their work this is proof that eqn. (3) is not a law: laws are dimensionally invariant.

That laws should have this property seems obvious to some and puzzling to others. Placing themselves in the latter group are Krantz et al. (1971, Section 10.10) who enumerate three reasons usually given for dimensional invariance (DI hereafter), calling them the "physical similarity", the "descriptive/deductive", and the "it couldn't be otherwise" arguments. Following Causey (1969) they consider the first in some detail (1971, Section 10.10.2), obtaining the result that each rational family of similar systems satisfies a dimensionally invariant law [1]. But they notice (1971, p. 512) that their result does not explain the DI of physical laws like eqn. (1) which describe a unique system (the universe). Therefore, this argument, while covering a lot of ground, is hardly decisive.

Again following Causey, Krantz et al. point out that the second argument is even weaker. This argument asserts that the fundamental equations of physics are DI, whence correct deductions from them must also be DI – thus accounting at most for the dimensional invariance of derived but not fundamental laws. This argument does, however, justify most of the applications of dimensional analysis; and with the first it greatly reduces the size of the puzzle.

The third argument, "it couldn't be otherwise", is based on the undoubted fact that the choice of units is arbitrary and the plausible but (to some) questionable assumption that correct descriptions of phenomena do not depend on arbitrary choices. Krantz, Luce, Suppes, and Tversky characterize this argument as an assertion of the belief that, if a satisfactory general definition of a physical law could be stated, it would imply DI (1971, p. 505).

This paper, without attempting a general definition of physical laws, offers two supports for the third argument. Both are undoubtedly known to many workers in the field but one of them has never to my knowledge been published. If they are somewhat pedestrian and commonsensical they nevertheless go beyond mere assertion, and enable us to see more clearly the implicit assumptions on which the "it could't be otherwise" argument is based. One support applies in the case of the empirical, and the other in the case of the numerical, interpretation of laws. If they are correct they push the problem back to a more fundamental level and thus reduce the puzzle by another small increment.

II. Two Interpretations of Laws

Let X be a set (e.g. of steel rods) on which are defined m weak orderings $\lesssim_1, \ldots, \lesssim_m$ determined by empirical comparisons (e.g. laying two elements of X side-by-side or placing them in opposite balance pans). Let $o_1, \ldots, o_m$ be empirical concatenations defined on X (e.g. laying two elements end-to-end or placing them in the same balance pan) such that for each $j = 1, \ldots, m$ and all $x, y, z, \in X$,

$$x o_j y \in X \text{ (closure)}$$

$$x o_j (y o_j z) = (x o_j y) o_j z \text{ (associativity)}.$$

Thus we have m ordered additive semigroups $G_j \equiv (X, \lesssim_j, o_j)$, each associated with a different empirical property of the elements of X (e.g. length or mass). The semigroups are defined on a single underlying set X, on the elements of which the empirical comparisons and operations are performed; but it is convenient, and common, to speak as if we were comparing or adding manifestations of a property – to say, for example, that $x\ o_1\ y$ is the sum of two lengths, instead of the more accurate, "$x\ o_1\ y$ is the concatenation of two elements with respect to length". We can then think of each semigroup as being associated with a different set (e.g. the "set of lengths"); this is permissible when we fix our attention on the properties rather than the elements themselves. We thus obtain m sets $X_j \equiv \{x_j, y_j, \ldots\}$, each being a property and the elements of which are the manifestations of that property in the underlying set X [2]. With $x_j \leqslant y_j$ meaning $x \lesssim_j y$ and $x_j + y_j$ meaning $x\ o_j\ y$, each X_j has the structure of an ordered additive semigroup. Note that $x_j + y_k$ is not defined for $j \neq k$.

We can introduce the definition $2x_j \equiv x_j + x_j$ and apply it recursively. Thus if there is a y_j equivalent (under the ordering $\lesssim_j$) to $x_j + x_j + x_j$ we can write $y_j = 3x_j$, and more generally $y_j = \alpha x_j$, where α is a real number. We can also define differences, products, exponents, and sequences in each X_j (Whitney, 1968), and from the interval topology determined by $\lesssim_j$ we get the notions of convergence and continuity.

There are several ways to define a multiplication on $X_1 \cup \ldots \cup X_m$. Suppose $u_j < x_j$ for all $x_j \in X_j - \{u_j\}$; then we can define $u_j * u_k$ as the unique element of X that is minimal with respect to both $\lesssim_j$ and $\lesssim_k$. Writing $x_j = \alpha_j u_j$ for all $x_j \in X_j$ and all $j = 1, \ldots, m$ we get $x_j * y_k = \alpha_j \alpha_k (u_j * u_k)$, and $*$ is a (multiplicative) commutative semigroup operation on $X_1 \cup \ldots \cup X_m$. For other approaches see Kurth (1965), Quade (1967), and Whitney (1968), who also provide the details that permit $X_1 \cup \ldots \cup X_m$ to be organized into a multiplicative vector space. Thus many of the familiar mathematical operations and concepts are applicable to empirical properties directly, not just through the medium of measurement.

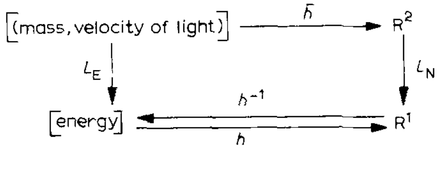
If the additive semigroup G_j has certain additional structure (for which see Krantz et al., 1971), it is homomorphic to the real numbers under a mapping h_j (one of infinitely many such), and $h_j(x)$ is the jth "property number" (e.g. length number, mass number) of the element x. To construct such a homomorphism is to measure the jth property (or, alternatively, to measure an element with respect to that property).

Armed with these notions we can distinguish between two interpretations of scientific laws. The interpretations ultimately come down to the same thing but intuitively they seem to be different, and they suggest different explanations of the DI principle. In the *empirical* interpretation a law is a functional relation between empirical properties, not numbers. A numerical expression of the law, however convenient it might be for calculating and reasoning, merely copies the fundamental relation between properties. The mathematical operations that can be performed on the property numbers are those and only those that can be performed on the properties themselves. (Please note that we use "empirical" in antithesis to "numerical", not "theoretical". In fact many of the empirical properties can be determined only by theoretical deductions, not observations – e.g. the mass of an electron, the utility of a commodity.)

In the *numerical* interpretation a law is a relation between numbers. The numbers are of two kinds, "dimensional" (say 3 inches) and "dimensionless" (say 3); but this distinction is only a shorthand way of keeping track of which numbers are measures and which are not; all the numbers are in fact real or complex. Two kinds of mapping occur – properties into numbers (the measurements) and numbers into numbers (the law). Once the numbers come into the picture they take over, as it were; operations on them are in no way restricted by operations permitted on properties.

In the numerical interpretation eqn. (1) is a mapping from R^2 into R^1: If the energy and mass of a particle and the velocity of light are represented by a coherent system of units then the energy number is proportional to the mass number, and the constant of proportionality is the square of the velocity number of light. In the empirical interpretation eqn. (1) is a mapping from $X_j \times X_k$ into X_i, where X_j, X_k, and X_i are respectively the mass, velocity, and energy semigroups: Energy is proportional to mass, and the constant of proportionality is the square of the velocity of light.

The interpretations are equivalent, as shown by the following diagram. (L_E is the law empirically interpreted, L_N the law numerically interpreted, and h and $\bar{h}$ are measurement homomorphisms. h^{-1} exists (in the absence of measurement error) because only one energy is associated with a given energy number as measured on a given scale.) But this equivalence does not prevent a difference of opinion about the legitimacy of operations on numbers.



III. The Empirical Interpretation

The empirical interpretation leads to a *positive explanation* of DI: "Like can only equal like". A mass can only equal another mass, a length another length, a density another density, etc. This proposition can be split into two parts, first that each property is an additive semigroup and second that the numerical expression of a law takes the form of an explicit function. The first says that if $x_j + y_j$ exists it belongs to the same additive semigroup as x_j and y_j, meaning in particular that x_j and y_j belong to the same semigroup and are thus manifestations of the same property. Hence the property numbers $h_j(x)$ and $h_j(y)$ can be added if and only if they represent the same property. It follows that each law expressed as a sum of terms, e.g. eqn. (2), is dimensionally homogeneous (each term has the same dimensions); and since dimensional homogeneity (DH) implies DI, the law is DI [3].

The first part thus recognizes a difference between the algebra of numbers and the algebra of properties. If w_j, x_j, y_k, and z_k are numbers then $w_j = x_j$ and $y_k = z_k$ imply $w_j + y_k = x_j + z_k$; but if they are manifestations of properties then $w_j + y_k$ is not even defined. Under the empirical interpretation Bridgman's example (1931, p. 42),

$$\left.\begin{aligned} v &= gt \\ s &= gt^2/2 \end{aligned}\right\} \rightarrow v + s = gt(1 + t/2)\,, \tag{4}$$

for many years a source of worry to students of dimensional analysis, is an improper application of numerical algebra to empirical properties [4]. Similarly,

$$E - p_1 q_1 = mc^2 + \Sigma_2^n p_i q_i\,, \tag{5}$$

derived from eqns. (1) and (2), would not follow from permissible operations in the algebra of properties [5].

The second part claims that each law expressed in the numerical form

$$F[h_1(x), ..., h_m(x)] = 0 \tag{6}$$

implicitly defines at least one mapping, say

$$h_1(x) = f[h_2(x), ..., h_m(x)] \,. \tag{7}$$

Then, though the first part does not reach eqn. (6), which need not be the sum of a number of terms and so need not be DH, in combination with the second part it reaches eqn. (6) through eqn. (7), the left and right sides of which must represent the same property. The second part must be true if the empirical interpretation is valid, as the hallmarks of that interpretation are that a law maps $X_1 \times ... \times X_{j-1} \times X_{j+1} \times ... \times X_m$ into X_j and that valid numerical expressions only copy the more fundamental relations between properties.

The question of the validity of the empirical interpretation is beyond our scope; but we see that it is a sufficient condition for the "it couldn't be otherwise" argument.

IV. The Numerical Interpretation

The numerical interpretation suggests a *normative* justification for DI: Without DI we could not know what the law says.

In this section we interpret eqns. (1)–(5) as numerical equations and the variables as numbers. It is convenient to call the variables by the names of the properties they represent, e.g. to call E "energy" instead of the "energy number". On the numerical interpretation eqns. (4) and (5) are perfectly valid expressions as all their variables are numbers (members of the same additive semigroup); and they are moreover true, as they follow from true laws. They are not DH, but they are DI. Changing the time unit in eqn. (4), for example, so that t is replaced by at, we obtain

$$s + v/a = (1 + at/2)gt/a, \quad a > 0, \tag{4a}$$

which holds for all values of v, s, g, and t satisfying eqn. (4) [6]. DH is sufficient, but not necessary for DI. Hence the considerations of the preceding section are not only unavailable; because they are devoted to showing the DH of laws, which is not necessary for DI, they are also unavailing: "not DH" does not imply "not DI". We want to see the consequences of "not DI".

But the consequences are obvious. Whatever a law says, it says it by its form; if its form changes when units are changed its message changes too. Since there is no way to decide which set of units is "correct" there is no way to know what the law says [7]. One or two examples should make this clear; but first, some additional notation.

Suppose n numbers $x_1, ..., x_n$ are related by a law. Let $x = (x_1, ..., x_n)$ and let $T: R^n \to R^n$ denote the transformation of the numbers that results from a change in units of measurement. In the simplest case, where the numbers are ratio scale measures, T is a linear transformation with a positive-definite diagonal matrix, the number of independent diagonal elements of which depend on the number of variables and the number of fundamental dimensions. For example, suppose x_1 is energy, with dimensions ML^2T^{-2}, x_2 is mass, with dimension M, and x_3 is the velocity of light, with dimension LT^{-1}; division of the mass unit by a, the length unit by b, and the time unit by k transforms (x_1, x_2, x_3) into

$$\left(\frac{ab^2}{k^2} x_1, ax_2, \frac{b}{k} x_3\right),$$

and T has the matrix

$$\begin{bmatrix} \frac{ab^2}{k^2} & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & \frac{b}{k} \end{bmatrix}, \quad a, b, k \text{ positive}.$$

A transformation of this kind, where the diagonal elements are limited only in the two respects of being positive and reflecting the dimensions of the variables, is called *permissible.*

For a law stated implicitly by eqn. (6) dimensional invariance is expressed as

$$F(x) = 0 \text{ iff } F[T(x)] = 0 \text{ for all permissible } T. \tag{8}$$

Now suppose a law lacked DI, e.g. suppose that, contrary to eqn. (1),

$$E = cm^2. \tag{9}$$

(9) is certainly true when the units of mass, length, and time are chosen so that $m = c$; it is then a true equation in numbers; but it is a false law.

It implies

$$\frac{dE}{dm} = 2cm = \text{a function of } m, \qquad (10)$$

where eqn. (1) implies

$$\frac{dE}{dm} = c^2 = \text{a constant.}$$

In this case a dynamical experiment could confirm the falsity of eqn. (10) and thence of eqn. (9). But dynamical experiments are not always feasible, especially in social science. It is therefore legitimate to assume them impossible in the present case. A physicist could still refute eqn. (9), by showing that it implies the wrong dimensional structure for energy (M^2LT^{-1} instead of ML^2T^{-2}), but this refutation assumes that laws are DH and therefore DI. As we are trying to see the consequences of "not DI" we cannot use the dimensional argument on eqn. (9).

Let us divide the mass unit by a, so that a mass formerly measured as x_2 becomes ax_2, and an energy that was x_1 becomes ax_1. Eqn. (9) no longer holds – immediate proof, to a believer in DI, that it is wrong. A nonbeliever would merely change the form of the expression to accommodate the changed measurements. Thus if in the initial units

$$x_1 = 27, \quad x_2 = 3, \quad x_3 = 3, \quad \text{i.e. } x = (27, 3, 3),$$

and we halve the mass unit so that $a = 2$, whence

$$T(x) = (54, 6, 3),$$

we must find a new functional form G such that

$$G[T(x)] = 0;$$

e.g.,

$$E = m(m + c). \qquad (11)$$

Eqn. (11) fits the data and is precisely as good a law as eqn. (9), but it has different implications. We don't know which set of implications to believe – i.e. we don't know what the law says.

If it should be objected that no choice between the implications is necessary, that each set of units carries its own implications which must only be kept straight, we can point to the existence of an infinite number of sets of units and hence an infinite number of differing implications which we can't examine: how can we know what the law says [8]? This objection clearly falls of its own weight.

The position is even clearer when a law cannot be directly tested by observation. Walras' Law eqn. (2), which holds for the entire economic system, can only in principle be tested directly, even in a static sense. Suppose, therefore, that instead of eqn. (2) Walras' Law were to assert eqn. (3). As we noted above, eqn. (3) holds in general economic equilibrium, and by a suitable choice of units it can be made to hold in a particular disequilibrium too; but it won't hold in that disequilibrium under all permissible transformations. Thus dividing the unit of the ith commodity by a_i gives

$$\Sigma_1^n p_i q_i(1 - a_i q_i) = 0, \tag{12}$$

which will not hold in the same disequilibrium state as eqn. (3). To maintain the truth of eqn. (3) we must change its form upon changing units of measurement. Since it is the form of the relation that expresses the law, we have to ask: What is the law? Is it eqn. (3) or (12)?

In sum, there is no sense in speaking of "the" law when it lacks DI. There are an infinite number of laws, generally contradictory, each holding in its own set of units, and carrying its own set of implications. Such "laws" are no help in understanding the world.

DI, on the other hand, *is* a help, not directly in understanding the world, but indirectly in showing up false ideas. Perhaps the best example of a positive use of this aid is Georgescu-Roegen's examination of the Marxian "proof" of the eventual breakdown of capitalism (Georgescu-Roegen, 1960). The proof depends, in part, on the following relation:

$$S = C + I + \frac{\mathrm{d}C}{\mathrm{d}t} \tag{13}$$

where S is surplus value, C is consumption expenditures of capitalists, I is investment in variable and fixed capital, and t is time [9]. Georgescu-Roegen criticizes this equation on the empirical interpretation: "As long as the letters in that formula stand for measurable material concepts and not for some Hegelian ideals, [C and $\mathrm{d}C/\mathrm{d}t$] cannot be added, any more than can *total* and *average* cost, for instance" (1960, p. 299). But the formula is just as meaningless on the numerical interpretation. S, C, and I are flows, with the dimensions money/time; if we divide the time unit by a we transform eqn. (13) into

$$\frac{S}{a} = \frac{C}{a} + \frac{I}{a} + \frac{1}{a^2}\,\frac{\mathrm{d}C}{\mathrm{d}t} \tag{14}$$

which is inconsistent with eqn. (13). If eqn. (13) were true its form would have to be changed upon changing units; the new form would say something different from eqn. (13). Which is correct? Marxian economics provides no answer, and indeed there is none.

DI helps in other ways as well. It acts as a *restriction* on the possible forms of laws and as such saves useless effort. It is like the laws of logic, which relieve us of the need to consider as possibly true descriptions of the world all possible statements that can be made. It thus acts to filter out obvious absurdities like eqns. (9), (11), (12), (13), and (14), allowing us to direct attention to more promising hypotheses. And a failure of DI, like a failure of the laws of logic, would remove all means of distinguishing between truth and falsehood, causing discourse – there could hardly be dialogue – to descend into solipsism. Is my proposed law refuted by ovservations? Never mind; I will simply change units until the law fits and dare you to prove me wrong. For if my law lacks DI, and if it is contradicted when I use a particular set of units, I can still make it fit the data as expressed in *some* set of units: If when changing units I can change the form of the law to fit the measurements, then I can *choose* units that give measurements which will fit the law.

Thus the normative justification for DI is that its absence would make quantitative science impossible.

V. On Definitions and Hypotheses

If the arguments of the preceding two sections are correct they apply to definitions and provisional hypotheses no less than to well-established laws. Bridgman (1931, Ch. 2) used the principle of dimensional invariance to explain why the "secondary quantities" of physics, e.g. density, are always defined as products of powers of the "primary quantities", e.g. mass and length. His demonstration was limited to the case where all the quantities are ratio scales; it was subsequently extended, with new results, to all the well-known scale types by Luce (1959) and Osborne (1970). The scale types and dimensions of the primary quantities place restrictions on the possible ways in which secondary quantities can be defined, and the restrictions are well-understood (they are summarized in Table 1 of Osborne, 1970). For example, an ordinal secondary quantity can be defined as a function of two or more primary quantities if and only if the latter are extensive quantities.

Nevertheless, the DI principle is sometimes violated – even by Samuelson, who more than anyone has acquainted the economics profession with the principle of invariance [10] (and Samuelson's errors tend to become doctrine). In his discussion of the "social welfare function", a secondary quantity $W(s)$ defined for state s as a function of the ordinal utility measures $u_i(s)$ of n individuals,

$$W(s) = F[u_1(s), ..., u_n(s)]\,, \tag{15}$$

Samuelson says:

> But the welfare function is itself only ordinally determinable so that there are an infinity of equally good indicators [i.e. ordinal measures] of it which can be used. Thus, if one of these is written as [(15)] and if we were to change from one set of . . . indexes of individual utility to another set [$(v_1, ..., v_n)$], we should simply change the form of the function F so as to leave all social decisions invariant (1947, p. 228) [11].

But this is mistaken. Changing the form of the function to make its ordering of social states conform, in a new set of individual utility units, to its ordering in the old set, amounts to prejudging the ordering. For if we can do that we can always find a form for the function and a set of utility units that yields any preassigned ordering to the states, regardless of individual preferences. The concept of social welfare then becomes purely private, and no amount of mathematical juggling can give it the least bit of objectivity.

Provisional hypotheses, too, must be expressed in dimensionally invariant form, or else they become subject to the kind of solipsistic "verification" noted above. But it is already obvious that the logical difference between hypotheses and laws is only one of degree, a matter of tests successfully passed; they are surely subject to the same rules of discourse. Thus Luce's proposal (1959, pp. 84–85) that DI be regarded as a principle of theory construction is wholly unexceptional, and his retraction (1962) in the face of criticism by Rozeboom (1962) unwarranted [12].

VI. A Note on Dimensional Constants

It is obvious that any relation can be made dimensionally invariant, even dimensionally homogeneous, by inserting enough dimensional constants. Eqn. (13) could be purified in this way by sticking a coefficient, of dimension time, on $\mathrm{d}C/\mathrm{d}t$. This seems to take the cutting edge off the principle of dimensional invariance.

It is true that the indiscriminate use of dimensional constants is not a way around the force of the principle of dimensional invariance in the case of secondary quantities (definitions). For example, suppose the social welfare function were defined as

$$W(s) = \Sigma_1^n k_i u_i(s) \tag{16}$$

where k_i is the weight given the utility of the ith individual in the computation of social welfare. By giving k_i the dimensions welfare/utility$_i$ we make eqn. (16) dimensionally invariant. But then consider the particular case of a "democratic" social welfare function, i.e. $k_1 = ... = k_n$. If we multiply the unit of the first person's utility by 3 and keep all other units unchanged the system of units is still coherent, for unless tastes are identical the utilities are not interpersonally comparable; but we must increase k_1 to $3k_1$, and the first person's weight in the social welfare computation will then increase threefold. It is true that the *numbers* $k_1 u_1$ and $3k_1 v_1$ are equal ($v_1 = u_1/3$) but that is irrelevant; the second unit chosen could equally well have been chosen instead of the old one for the expression (15) [13]. In other words there is no sense in speaking of equal weights, or indeed of any particular scheme of weights, in summed weighted values of unlike quantities. This proposition extends immediately to all mathematical forms of the definitions of secondary quantities.

But the situation appears to be different in the case of laws. In this case we are not inventing new quantities but trying to discover the relation between given ones, and there is no logical reason why such relations are not full of dimensional constants.

Bridgman (1931, Ch. 5) has argued, convincingly in my view, if not completely rigorously, that the number of dimensional constants in a derived relation cannot exceed the number in the fundamental relations from which it is derived, and that, in turn, the number of dimensional constants in a fundamental relation cannot exceed the number of independent variables which it relates. This result is most valuable in those cases – rare in social science – where we know the fundamental relations. It is also helpful in some cases where we don't know the form of the underlying relations but do know what variables they relate. For example, an individual's demand x_1 for commodity 1 is a function of his income y and the prices p_i of all commodities ($i = 1, ..., n$); this function is derived from two fundamental relations, the utility function

$$u = \phi(x_1, ..., x_n) \tag{17}$$

and the income constraint

$$\Sigma_1^n p_i x_i = y\,. \tag{18}$$

The constraint has no dimensional constants, the utility function at most n (if Bridgman is correct); hence the demand function

$$x_1 = D(p_1, \ldots, p_n, y), \tag{19}$$

which is derived by maximizing eqn. (17) subject to eqn. (18), has at most n dimensional constants. We can conclude immediately that the linear form of eqn. (19),

$$x_1 = \alpha + \Sigma_1^n \beta_i p_i + \gamma y$$

cannot possibly be true, for it contains $n + 2$ dimensional constants (α, γ, and β_i for $i = 1, \ldots, n$).

But Bridgman's argument does not help in all cases. Thus a competitive firm producing commodity 1 has a supply function

$$x_1 = f(p_1), \tag{20}$$

determined by maximizing its profit P,

$$P = p_1 x_1 - \Sigma_1^m w_i v_i \tag{21}$$

where v_i is the rate of input of the ith factor of production and w_i is its price, subject to the production function

$$x_1 = g(v_1, \ldots, v_m). \tag{22}$$

The constraint has at most m dimensional constants, the objective function none; hence in principle eqn. (2) could have m dimensional constants, and none of its economically possible forms can be eliminated by dimensional reasoning.

Thus the real force of the principle of dimensional invariance, and the usefulness of dimensional analysis, depends on the number of dimensional constants in the fundamental laws. In physical science no known fundamental law contains more than one such constant, or so it appears to a layman. Is this an inherent characteristic of physical science? of fundamental laws? of neither? We can only hope that the second is true. But on what lines should this question be approached?

Notes

[1] Examples of similar systems are given by Bridgman (1931, Ch. 7). All simple pendulums are similar in the relation between length L and period T: $T = k\sqrt{L/g}$, where k is a dimensionless constant and g is the acceleration of gravity. Thus simple pendulums form a family of similar systems. And the family is rational because the dimensions of time and length occur in rational exponents. Bridgman, incidentally, appears to have been the first to wonder why laws are dimensionally invariant (1931, p. 13).

[2] The X_j are usually called dimensions and their elements are called physical quantities.

[3] If an expression is DH a (coherent) change of units causes each of its terms to be multiplied by the same conversion factor, which therefore cancels out to leave the expression unchanged in form.

[4] Bridgman fails to distinguish clearly between the interpretations. In some places he uses a symbol "interchangeably for the quantity itself and for its numerical measure", (1931, p. 38) and in others only for the numerical measure (e.g. 1931, p. 41).

[5] It is possible that a more satisfactory theory of physical quantities would distinguish between zeroes. $E-mc^2$ is zero energy, $\Sigma_1^n p_i q_i$ is zero net expenditure. Perhaps zero energy should be treated as a different thing from zero net expenditure. How this could be accomplished is not at all clear.

[6] The dimensions of v are length/time (i.e. LT^{-1}) and those of g are length/(time)^2 (i.e. LT^{-2}). As Bridgman notes (1931, p. 41), DI holds because eqn. (4a) is not the only relation between the variables.

[7] Some units are better than others for computations. That has nothing to do with the principle that, from a scientific standpoint, units of measurement can only be chosen arbitrarily.

[8] That an expression lacking DI has as many sets of implications as there are sets of measurement units, most of the sets of implications being contradictory, must be distinguished from the fact that a true law also has an infinite number of implications, all of which, however, are mutually consistent. And it must be distinguished from the general problem of scientific inference – the inescapable problem that an infinite number of hypotheses are consistent with a given set of data.

[9] Eqn. (13) is Georgescu-Roegen's eqn. (3^{bis}) in a different notation.

[10] See especially *Constancy of the Marginal Utility of Income* (Samuelson, 1942) and the section of the same title in Samuelson, 1947 (pp. 189–195).

[11] Where the ellipsis occurs Samuelson has the word "cardinal". However, he clearly is not assuming that utility is cardinally measurable; rather, he is working with ordinal utility but here as elsewhere calls the resulting numbers "cardinal indexes", letting the word "index" indicate the ordinal (uniqueness up to increasing transformation) of the numbers. Samuelson did not, of course, have the advantage of the modern clarifications in the theory of measurement, and his largely confused discussion of the social welfare function (1947, pp. 219–228) suffers accordingly. Unfortunately it has formed the views of many economists.

[12] Rozeboom's criticism amounted to this: Any hypothesis lacking DI can be given it by inserting enough dimensional constants. But this implicitly accepts the principle of DI.

[13] Ordinal quantities do not, of course, have "units", but for expositional convenience we can speak of a change in units to mean a linear transformation, which is merely a special case of the increasing transformations up to which ordinal quantities are numerical.

References

Bridgman, P.W. (1931). *Dimensional Analysis,* Revised ed. New Haven: Yale University Press.

Causey, R.L. (1969). "Derived Measurement, Dimensions, and Dimensional Analysis," *Philosophy of Science*, 36: 252–70.

Georgescu-Roegen, N. (1960). "Mathematical Proofs of the Breakdown of Capitalism," *Econometrica*, 28: 225–43.

Krantz, D.H., Luce, R.D., Suppes, P. and Tversky, A. (1971). *Foundations of Measurement*, Vol. I. New York: Academic Press.

Kurth, R. (1965). "A Note on Dimensional Analysis," *American Mathematical Monthly*, 72: 965–69.

Luce, R.D. (1959). "On the Possible Psychophysical Laws," *Psychological Review*, 66: 81–95.

Luce, R.D. (1962). Comments on Rozeboom's Criticisms of "On the Possible Psychological Laws," *Psychological Review*, 69: 548–51

Osborne, D.K. (1970). "Further Extensions of a Theorem of Dimensional Analysis," *Journal of Mathematical Psychology*, 7: 236–42.

Quade, W. (1961). "Über die algebraische Struktur des Grössenkalküls der Physik," *Abhandlungen der Braunschweigischen Wissenschaftlichen Gesellschaft*, 13: 24–64. English translation in F.J. de Jong, *Dimensional Analysis for Economists*. Amsterdam: North-Holland Publishing Co., 1967.

Rozeboom, W.W. (1962). "The Untenability of Luce's Principle," *Psychological Review*, 69: 542–47.

Samuelson, P.A. (1942). "Constancy of the Marginal Utility of Income," in O. Lange, ed., *Studies in Mathematical Economics and Econometrics*. Chicago: University of Chicago Press.

Samuelson, P.A. (1947). *Foundations of Economic Analysis*. Cambridge: Harvard University Press.

Whitney, A., "The Mathematics of Physical Quantities. Part I: Mathematical Models for Measurement. Part II: Quantity Structures and Dimensional Analysis," *American Mathematical Monthly*, 75: 115–38, 227–56.