Triggered Correlation

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Abstract—**This paper shows to what extent an average-response computer can be utilized for computing a cross-correlation function. This type of computer needs synchronization pulses, and the simplest methods of computation are those in which these pulses are directly derived from one of the signals (triggered correlation). The first method is to generate a synchronization pulse whenever the signal crosses a pre-set threshold in any direction. In this case, the computer output function is shown to be proportional to the true correlation function, for Gaussian signals. In a second method, synchronization pulses are produced when the signal crosses the threshold in a specified (e.g., positive) direction. Then the computer output is found to be contaminated by a systematic error, which, in turn, depends on the derivative of the correlation function. These two methods are described in detail, both with respect to the results and to the accuracy obtainable. Several other, less important, methods are only briefly described.**

The second method (single-direction triggered correlation) is the simplest and most attractive one. Its feasibility is shown by means of a practical example. It is also useful in the analysis of compound systems, namely, systems that consist of a linear circuit followed by a triggering and pulse-forming circuit. In this respect, the method can be, and, as a matter of fact, has been applied to problems like the excitation of nerve impulses in sensory organs. Such an application is briefly described.

Introduction

AUTO-CORRELATION and cross-correlation functions are often used to determine the parameters of linear circuits under random excitation [1]. The computation of these functions generally requires a good deal of instrumentation. It is then logical that simplifications of the procedure have been sought. These met with considerable success: in many cases signals can be distorted quite heavily before their correlation is irretrievably lost [2]. Either one or both of the two input signals can be infinitely clipped, for instance: the resulting correlation function retains its general shape.

For the solution of other problems, special-purpose computers (average-response computers) have been made available that perform a specialized type of correlation computation [3], [4]. These instruments operate, in fact, on one signal, but they need to be triggered by a series of synchronization pulses. They determine the average waveform during a specified period after the synchronization pulses. They have become quite popular, especially in certain fields of physiology.

The purpose of this paper is twofold. First, it will be shown that an average-response computer can be utilized to obtain a cross-correlation function in the general case. The success of the method depends, of course, on the method used for providing the proper triggering pulses; quite simple schemes of triggering will prove fruitful. The second aspect of "triggered correlation" lies in its use in the analysis of compound nonlinear systems. A system consisting of a linear filter followed by a triggering circuit and a pulse generator can be analyzed by application of one of the triggering schemes described.

Manuscript received July, 1967; revised December 11, 1967.
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Practical application of the method is illustrated by typical experimental results. These cover two aspects: general cross-correlation computation and analysis of nonlinear systems (in this case, the excitation of nerve impulses in the inner ear).

Simplified Correlation Procedures

According to the expression (1) for the finite-time estimate $\phi_{xy}^*(\tau)$ of the cross-correlation function $\phi_{xy}(\tau)$:

$$\phi_{xy}^*(\tau) = \frac{1}{T}\int_0^T x(t+\tau)y(t)dt, \tag{1}$$

the following operations must be executed:

1) delay of one signal with respect to the other
2) multiplication of two signal values
3) integration or averaging.

The procedure is eminently suited to a general-purpose computer, although special measures must often be taken to handle a large amount of input data.

Several methods have been advocated to reduce demands on processing. In *relay correlation* [5], one of the signals is infinitely clipped; i.e., it is replaced by $+1$ when it is positive and by -1 when it is negative. Either of the two signals $x(t)$ or $y(t)$ can be treated in this manner:

$$\begin{aligned} \phi_{xyr}^*(\tau) &= \frac{1}{T}\int_0^T x(t+\tau)\,\mathrm{sgn}\,\{y(t)\}dt \\ \phi_{rxy}^*(\tau) &= \frac{1}{T}\int_0^T \mathrm{sgn}\,\{x(t+\tau)\}y(t)dt. \end{aligned} \tag{2}$$

In general, the relay correlation function (RCF) resembles the true correlation function surprisingly well. For Gaussian signals with zero means, it is even proportional to the true correlation function. Appendix I gives a condensed derivation of this property, and, in addition, gives the results of the general not-centered case. We will need these results later.

The second method used to reduce the amount of information processing is a more drastic one. Both the signals $x(t)$ and $y(t)$ are infinitely clipped. The resulting *polarity correlation function* (PCF):

$$\phi_{xyp}^*(\tau) = \frac{1}{T}\int_0^T \operatorname{sgn}\{x(t+\tau)\}\cdot\operatorname{sgn}\{y(t)\}\,dt, \quad (3)$$

again is a good substitute for the true correlation function [2], [5], [6]. (See (15) of Appendix I.) Despite the extremely crude signal representation the properties of the original correlation function are well retained. One can conclude that a pair of random signals really can incur a great deal of distortion before their mutual correlation is lost. Other schemes of nonlinear signal processing might turn out to be successful as well.

Average-response computer is the common name for special-purpose computers designed for retrieving hidden signals from superimposed noise. These also carry out a kind of simplified correlation [3], [4], [7]: they compute the average value of the input signal $x(t+\tau)$ taken over a large number of instants $t=t_i$ ($i=0, 1, \cdots N$) at which a synchronization impulse is given. The output is computed for a large number (400 to 1000) of values of τ; the results are usually presented on an oscillograph as a function $\phi(\tau)$ of τ. As a matter of fact, an average-response computer adds up all the signal fragments $x(t_i+\tau)$; the division by N is tacitly assumed to be taken care of by the interpretation of the result.

An average-response computer can be used to retrieve a hidden periodic signal from noise, and also to detect responses time-locked to stimuli given at random instances. In both modes, the average-response computer has been extremely useful in fields like general physiology, neurophysiology, biophysics, EEG-analysis, etc. [3], [4].

One may ask whether such an instrument as the average-response computer may be utilized to carry out a general correlation computation. Then we start with two functions, the x- and the y-signal. One of these, e.g., $x(t)$, can be handled in the normal way; the other one has to be converted into a series of sync pulses. The simplest way to do this is to let the sync pulse be triggered whenever the signal $y(t)$ satisfies a certain condition. We may name such a method *triggered correlation*, since the correlation procedure is triggered directly by one of the signals. Several cases of interest can be distinguished.

1) Dual-polarity triggered correlation. Here the sync pulse is generated when $y(t)$ crosses a pre-set threshold b, irrespective of the direction of crossing (see Fig. 1).
2) Single-polarity triggered correlation. The sync pulse is generated when $y(t)$ crosses the threshold b in a specified, e.g., the positive, direction (see Fig. 2).
3) Extreme-value triggered correlation. The sync pulse is generated when $y(t)$ passes through a maximum (see Fig. 3).

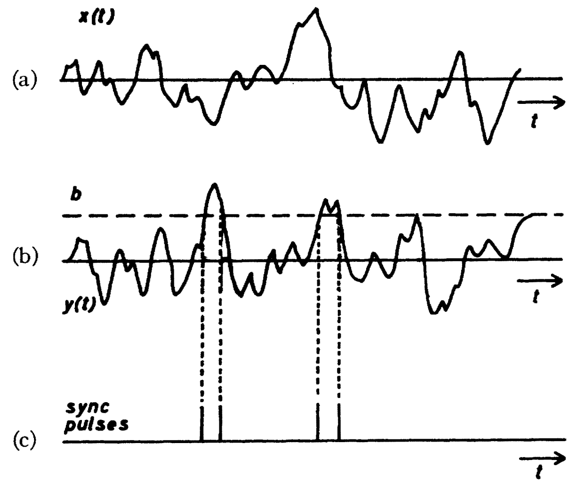

Fig. 1. Dual-polarity triggering. (a) x-signal, to be processed. (b) y-signal and threshold level b. (c) Series of sync pulses.

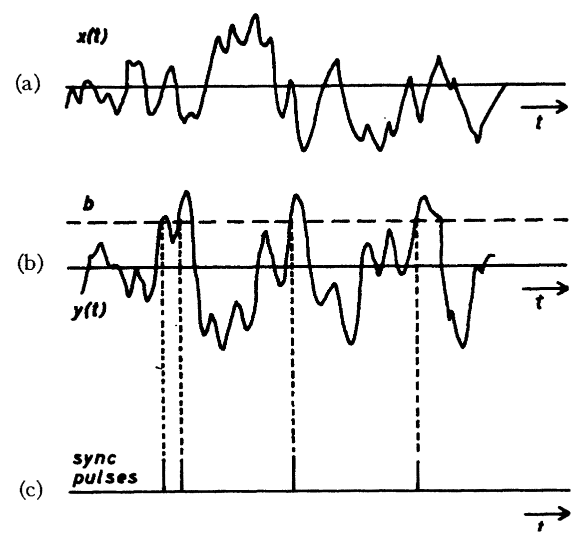

Fig. 2. Single-polarity triggering. (a) x-signal, to be processed. (b) y-signal and threshold level b. (c) Series of sync pulses.

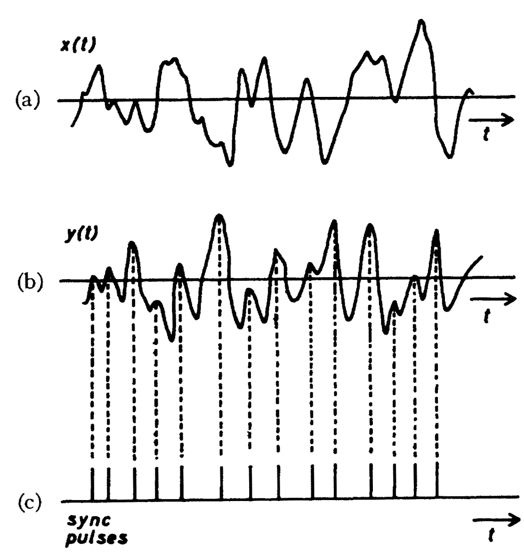

Fig. 3. Maximal-value triggering. (a) x-signal, to be processed. (b) y-signal. (c) Series of sync pulses.

4) Quasi-relay triggering. In this method, triggering impulses are synchronous with a clock and occur whenever $y(t)$ is positive.

We will discuss the character and the accuracy of these methods in some detail. We may predict that method 2), being the simplest one, will be the most important one in practice. That is one reason why this method is elaborated upon somewhat more than the others.

The list of possibilities given above is by no means exhaustive. There are many other ways to incorporate an average-response computer in a correlator. Those methods may utilize, for instance, a separate electronic signal multiplier. In that case, the computer is conveniently employed as a multiaddress storage and integration device. Or else, in method 2) an auxiliary signal may be added to one of the signals under study in order to distribute the triggering moment uniformly over the range of signal values [6]. Such methods will give results that are directly comparable to those from true correlation computation since they tend to simulate relation (1) or its ensemble-average counterpart directly. For the purpose of this paper, however, such highly perfected procedures as these are not interesting. We will treat here only the simplest procedures like the ones tabulated above. We will find that these give, in general, a very useful approximate evaluation of the correlation, at least for Gaussian signals. The closeness of the obtained result to the exact function may be surprising at first sight. On closer inspection it appears as a logical consequence of the fact that correlation functions are highly resistant to nonlinear processing of the signals [2]. The triggering procedure employed is then just one specific kind of nonlinear no-memory processing.

General Theory

Case 1. Dual-Polarity Triggered Correlation

In this case, the computer is triggered when one of the signals, e.g. the y-signal, crosses a threshold b, irrespective of the direction of crossing. Consider two signals $x(t)$ and $y(t)$, of which the cross-correlation function $\phi_{xy}(\tau)$ is to be determined. For a given, fixed value of τ, the signal values can be regarded as two random variables. Let us assume that these variables, for convenience called x and y instead of $x(t)$ and $y(t+\tau)$, have a joint Gaussian probability density

$$p(x, y) = c \exp \left\{-\frac{x^2 - 2\rho xy + y^2}{2(1-\rho^2)}\right\} \tag{4}$$

with

$$c = \frac{1}{2\pi(1-\rho^2)^{\frac{1}{2}}}.$$

The variances of x and y are both unity, and the correlation coefficient is ρ. We use ρ for reasons of simplicity instead of the value of the correlation function $\phi_{xy}(\tau)$ for the pertinent value of τ.

We desire to compute the average value $\bar{x}_b$ of x under the condition that y equals b, irrespective of the direction in which this threshold is crossed. The conditional probability $q(x|y=b)$ is simply obtained by putting $y=b$ into (4)

$$q(x \mid y = b) = kc \exp \left\{-\frac{x^2 - 2\rho bx + b^2}{2(1-\rho^2)}\right\}.$$

The constant k is included to ensure that the total probability, integrated over all x-values, becomes unity. Then

$$k = \sqrt{2\pi} \exp (b^2/2).$$

The average value $\bar{x}_b$ of x under the probability density function $q(x|y=b)$ becomes

$$\bar{x}_b = \int_{-\infty}^{+\infty} xq(x \mid y = b)dx = \rho b. \tag{5}$$

Furthermore, the variance σ_x^2 of each sample of x under this condition is $(1-\rho^2)$. It is this value $\bar{x}_b$ that the average response computer ultimately produces at each of its addresses when it is triggered at all the instants t_i that $y(t_i)=b$ and averages the signal $x(t_i+\tau)$. If we visualize $\bar{x}_b$ next as a function $\bar{x}_b(\tau)$ of τ, we proved that the computed function $\bar{x}_b(\tau)$ is proportional to the true correlation function $\phi_{xy}(\tau)$. The constant of proportionality is b.

Case 2. Single-Polarity Triggered Correlation

Assume now that the average-response computer is triggered when $y(t)$ crosses the pre-set threshold level b only in the positive direction, as in Fig. 2. The statistical properties of $x(t_i+\tau)$ at those instants t_i cannot be obtained any more from the two-dimensional probability density function (4). We must take a third variable into account, the time derivative of $y(t)$

$$z(t) = \alpha \frac{d}{dt} y(t).$$

Here a is a positive constant introduced for the purpose of normalization. The average-response computer now produces the average value of $x(t_i+\tau)$, averaged over those instants t_i that $y(t_i)=b$ and $z(t_i)$ is positive. We thus have to consider three signals $x(t)$, $y(t)$, and $z(t)$ all normalized to have unity variance with three cross-correlation functions $\phi_{xy}(\tau)$, $\phi_{xz}(\tau)$, and $\phi_{yz}(\tau)$. The correlation function $\phi_{xz}(\tau)$ is proportional to the derivative of $\phi_{xy}(\tau)$ with respect to τ

$$\frac{d}{d\tau}\phi_{xy}(\tau) = \frac{d}{d\tau}E\{x(t)y(t-\tau)\} = E\left\{x(t)\frac{d}{d\tau}y(t-\tau)\right\}$$

$$= -E\left\{x(t)\frac{d}{dt}y(t-\tau)\right\} = -\phi_{xz}(\tau)/a.$$

Similarly,

$$\phi_{yz}(\tau) = -a\frac{d}{d\tau}\phi_{yy}(\tau).$$

For a certain value of τ, the signal values $x(t+\tau)$, $y(t)$, and $z(t)$ can be taken as random variables. Let us call these x, y, and z. They have a joint Gaussian probability distribution and their correlation coefficients ρ_{xy}, ρ_{xz}, and ρ_{yz} are

$$\rho_{xy} = \phi_{xy}(\tau)$$

$$\rho_{xz} = \phi_{xz}(\tau) = -a\frac{d}{d\tau}\phi_{xy}(\tau)$$

$$\rho_{yz} = \phi_{yz}(0) = -a\frac{d}{d\tau}\phi_{yy}(0).$$

The last relation is due to the fact that y and z are to be taken at the same value of t. For an auto-correlation function like $\phi_{yy}(\tau)$, the derivative at $\tau=0$ is zero, provided the associated spectral density function $\Phi_{yy}(\omega)$ decreases fast enough toward zero for $\omega\rightarrow\infty$. We assume that the latter condition is fulfilled; the correlation ρ_{yz} to be used in the calculations is then zero.

The desired average value $\bar{x}_+$ of x, subject to the conditions $y=b$ and $z\geq 0$, can be computed in several ways. One is the method discussed in the preceding section. This method, based on the probability density function, will yield unwieldy intermediate results, despite the fact that one correlation coefficient is zero. We used another method based on the characteristic function. The derivation is given in Appendix II. The result is again valid for Gaussian signals

$$\bar{x}_+ = b\rho_{xy} + \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\rho_{xz}. \tag{6}$$

The first term is the same as the one obtained in case 1. It indicates an average that is proportional to the desired correlation value. The second term is to be regarded as a correction; it represents the systematic error made by considering only one-way threshold crossings.

Case 3. Maximal-Value Triggered Correlation

The properties of this method are easily derived from those of the second method. If one chooses the triggering instants at the extrema of $y(t)$, one effectively triggers at the zeros of $\dot{y}(t)$ the time derivative of $y(t)$. If only maxima are counted, the value of $\ddot{y}(t)$, the second derivative, is always negative. We thus have to replace $y(t)$ by $a\dot{y}(t)$ and $z(t)$ by $-a^2\ddot{y}(t)$ in the above derivation. The result reads (since the equivalent threshold level b is zero)

$$\bar{x}_+(\tau) = -\left(\frac{2}{\pi}\right)^{\frac{1}{2}}a^2\frac{d^2}{d\tau^2}\phi_{xy}(\tau). \tag{7}$$

The method of auto- and cross-*relation*, described by Kamp *et al.* [8] is very similar to this case. In their method, an average-response computer is triggered at moments that are slightly differently defined. The y-signal can be written as a sum of (random) Fourier components [9]

$$y(t) = \sum_{n=1}^{N} a_n \cos(\omega_n t + \psi_n)$$

where a_n and ψ_n are random variables, a_n with a Gaussian distribution, ψ_n with a uniform distribution (between 0 and 2π), and $\omega_n = n2\pi f_0$. This signal is supposed to be band-limited, hence N is finite. Define the Hilbert transform $\hat{y}(t)$ of $y(t)$ as the signal with all components shifted 90° in phase

$$\hat{y}(t) = \sum_{n=1}^{N} a_n \sin(\omega_n t + \psi_n).$$

Now the combination $\eta(t)=y(t)+i\cdot\hat{y}(t)$ is the so-called "analytic signal" or pre-envelope signal [10] which can be characterized by an amplitude $|\eta(t)|$ and a phase function arc tan $\{\hat{y}(t)/y(t)\}$. In the *relation* procedure, triggering occurs when $\eta(t)$ passes a pre-set phase. If we choose this phase zero, triggering will occur at the moments t_i where

$$\hat{y}(t_i) = \sum_{n=1}^{N} a_n \sin(\omega_n t_i + \psi_n) = 0$$

and

$$\frac{d}{dt}\hat{y}(t_i) \leq 0.$$

The moments t_j of maximum $y(t)$, on the other hand, are given by

$$\dot{y}(t_j) = -\sum_{n=1}^{N} a_n\omega_n \sin(\omega_n t_j + \psi_n) = 0$$

and

$$\ddot{y}(t_j) \leq 0.$$

If the y-signal is a narrow-band signal, one may replace the coefficient ω_n in the latter expression by the average frequency ω_0 and these two equations then become identical. Hence, when this provision is met, triggering occurs approximately at the maxima of $y(t)$. Then the method produces the second derivative of the true correlation function $\phi_{xy}(\tau)$. For narrow-band signals, this result resembles the correlation function very much.

In the most general case the *relation* method yields a result that is proportional to the derivative of $\hat{\phi}_{xy}(\tau)$ (the latter function being the Hilbert transform of the original correlation function) with respect to τ. In any event, the *relation* method of Kamp *et al.* is an example of a triggered correlation procedure, and as such, it is one of the many possible correlation methods.

Case 4. Quasi-Relay Triggered Correlation

In this case, trigger pulses are generated synchronous to a clock whenever $y(t) \geq 0$. During the time $y(t)$ stays positive several trigger pulses can be present, and each must start a computer run. This is difficult to realize. The theory of this method, when the clock frequency is high enough, is equivalent to that of relay correlation. We refer to Appendix I. For low clock rates, one incurs the same problems as in any sampled-data system with too low a clock rate. A discussion of this point would be far outside the scope of this paper.

Discussion, Technical Aspects, Accuracy

The point, briefly mentioned in Case 4, is of interest in all cases. The action of an average-response computer is such that on each received sync pulse the whole range of τ-values is processed. Many sync pulses, therefore, tend to be lost. When enough data are available, this does not matter. In the other case we can solve the problem, in principle, by repeating the signals again and again, taking care that each time new waveform fragments are processed. We will tacitly assume that this problem is solved anyhow, in other words, that the maximal information content of the signals is available. We then have to compare the methods as to the efficiency with which the information is utilized, i.e., with respect to their systematic and random errors.

Case 1. Dual-Polarity Triggered Correlation

The first method, although technically not the simplest one, appears to be the best one for computation of a correlation function with an average-response computer. When technical problems concerning a possible difference of threshold levels for the two polarities are solved, this method yields the correct answer. This property in itself is independent of the height of the threshold level b. If we look at the accuracy for finite-time observation, however, the triggering level does play a role. Choosing a very high b-value gives a good signal output, but triggering does not occur very often, so that accuracy is poor.

In order to put this into a more quantitative form we assume that $y(t)$ is Gaussian noise with unity variance, and has a uniform spectral density $\Phi_{yy}(\omega) = 1/4\pi W_0$ up to a frequency $\omega = 2\pi W_0$ and zero beyond. The average number of times per second that the threshold b is passed is in general [11]

$$\overline{N}_b = \frac{1}{\pi} \cdot \bar{\omega}_0 \cdot \exp\left(-\frac{b^2}{2}\right)$$

where $\bar{\omega}_0$ is the root-mean-square angular frequency defined as

$$\bar{\omega}_0 = \left[\frac{\int_0^\infty \omega^2 \Phi_{yy}(\omega) d\omega}{\int_0^\infty \Phi_{yy}(\omega) d\omega}\right]^{\frac{1}{2}}.$$

In our case

$$\bar{\omega}_0 = \frac{2\pi}{\sqrt{3}} W_0.$$

In T seconds there will be on the average $\overline{N}_b T$ computer runs. In the case $x(t)$ and $y(t)$ are uncorrelated, and $x(t)$ is white noise, the computer processes independent signals. The output, after averaging, will have a variance σ_n^2

$$\sigma_n^2 = \frac{1}{\overline{N}_b T}.$$

If now the measured correlation ρ is small ($\rho^2 \ll 1$), the answer $\bar{x} = b\rho$ will be contaminated by a noise with a variance nearly equal to σ_n^2. Then the signal-to-noise ratio (which in effect determines the smallest correlation value that is discernible) is

$$A_{sn} = \rho^2 b^2 \overline{N}_b T = \frac{2}{\sqrt{3}} W_0 T b^2 \rho^2 \exp(-b^2/2). \quad (8)$$

This expression is maximal for $b = \sqrt{2}$, and assumes then a value

$$A_{sn}^{opt} = 0.849\, W_0 T \rho^2. \quad (9)$$

Under comparable circumstances, the signal-to-noise ratio for "true correlation" is [12]

$$A_{sn}^{max} = 2 W_0 T \rho^2. \quad (10)$$

Hence, dual-polarity triggered correlation produces for small ρ, an output signal-to-noise ratio at least 3.7 dB lower than true correlation. For the same accuracy 2.35 times as many data must be processed. This loss is of the same order of magnitude as that for polarity correlation [12].

Case 2. Single-Polarity Triggered Correlation

Single-polarity triggering is the simplest method of using an average-response computer to obtain a cross-correlation function. It requires a simple trigger circuit that produces a sync pulse whenever the y-signal passes the pre-set threshold b in a specified, e.g., the positive, direction. The theory has shown that this method yields a linear combination of the desired correlation function $\phi_{xy}(\tau)$, and a disturbing term proportional to $\phi_{xx}(\tau)$. This disturbing term in (6) is a systematic error. Since there is in general no a priori reason why $\phi_{xx}(\tau)$ should be materially smaller than $\phi_{xy}(\tau)$, it can only be made small in proportion by choosing a high value of b. Or else this systematic error can be removed by cancellation. One processes the signals twice, with different values of b, and subtracts the results. In this way only the main term, the one proportional to ρ_{xy}, will be left.

This procedure can even be carried out when the amount of available signal data is limited so that one should worry about random errors. For a single processing run, accuracy is highest when b equals $\sqrt{2}$, as before. The output signal-to-noise ratio then is 3 dB lower than in Case 1 since the number of computer runs is halved. In this, the most simple of all cases of triggered correlation, the signal-to-noise ratio thus is 6.7 dB worse than for normal correlation. Accuracy will be 3 dB better when one carries out the cancellation procedure with a second processing run, this time with $b = -\sqrt{2}$. The situation is then fully comparable with the one of Case 1.

In practice there is one pitfall: the computer effectively does not produce an average, but a sum. Care must therefore be taken to make the number of computer runs the same in the two processing runs.

Case 3. Maximal-Valued Triggered Correlation

The average number of maxima of $y(t)$ is equal to the average number of zero crossings of the derivative $\dot{y}(t)$ in a specified direction. For a y-signal with uniform power spectrum $\Phi_{yy}(\omega)$ over the range $-2\pi W_0 < \omega < 2\pi W_0$, the average number $\overline{N}_m T$ of maxima in T seconds is [11]

$$\overline{N}_m T = \left(\tfrac{3}{5}\right)^{\frac{1}{2}} W_0 T.$$

For small values of $\rho_{x\ddot{y}}$, the signal-to-noise ratio is

$$A_{\mathrm{sn}} = \frac{2}{\pi}\left(\frac{3}{5}\right)^{\frac{1}{2}} W_0 T \rho_{x\ddot{y}}^2 = 0.494\ W_0 T \rho_{x\ddot{y}}^2.$$

If one would compute the correlation $\rho_{x\ddot{y}}$ directly, the signal-to-noise ratio would again be $2WT\rho_{x\ddot{y}}^2$. Hence, the maximal-value triggering method is approximately 6.1 dB worse than true correlation; it requires 4.1 times as many data for the same signal-to-noise ratio. In this respect, it is entirely comparable to method 2 with a loss of 6.7 dB.

Case 4. Quasi-Relay Triggered Correlation

Since Case 4 can hardly be realized in practice, we refrain from discussing its accuracy in details. Let it be sufficient to mention that the maximal loss of relay correlation in signal-to-noise ratio is approximately 2 dB [12]. Quasi-relay correlation is 3 dB worse.

A Practical Example (*Case 2*)

As an example to illustrate the method, we chose a case where the shape of the correlation function can be predicted. When a linear network is fed with white noise, the input–output cross-correlation function $\rho_{xy}(\tau)$ has the same shape as the network's impulse response, $h(\tau)$ but with τ reversed [1]

$$\phi_{xy}(\tau) = h(-\tau).$$

Hence, for a given network, the correlation function is known and the actual result of the computation can easily be compared with the true answer. We chose this property as the basis for our proof of the feasibility of the method. We measured across a bandpass filter of one octave bandwidth. We called the input $x(t)$ and the output signal $y(t)$, and then computed the correlation function $\phi_{xy}(\tau)$ by letting $y(t)$ trigger the computer. Since $y(t)$ could only be correlated with the past of $x(t)$, we had to delay $x(t)$ before it was fed to the signal input of the computer.[1] The result of a computer run is shown in Fig. 4, together with the network's impulse response measured directly. The correlogram must be read from right to left, and it is seen to correspond well with the impulse response. To illustrate this agreement somewhat further, the frequency spectrum associated with the measured correlation function was determined. With the computer, after the run, in the "display" mode a repetitive signal was fed to the deflection plates of the computer's oscilloscope. This signal was analyzed by a spectrum recorder (Rohde and Schwarz type FNA). Due to the periodicity, the spectrum showed peaks at multiples of the scanning frequency (approximately 64 Hz in this case). See Fig. 5. The envelope of

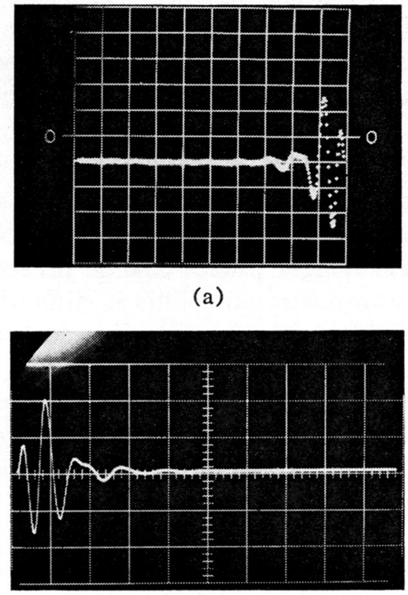
(a)
(b)

Fig. 4. (a) Result of a computer run (see text). The input–output correlation of an octave bandpass filter with white-noise excitation is measured. Correlation function to be read from right to left. Computer: Nuclear–Chicago type 7100 Data Retrieval Computer. Filter: Wandel and Goltermann OB 5, octave filter, set at a passband of 560 to 1120 Hz. Root-mean-square value of y-signal: 0.21 volt. Threshold level: 0.60 volt. Number of sync pulses processed: 10 000. Analysis time (full-width screen): 15 ms. (b) Impulse response of the filter as photographed directly from Tektronix type 535-A Oscilloscope.

[1] Otherwise, $x(t)$ and $y(t)$ can be stored on magnetic tape and reproduced backward so as to invert the direction of t.

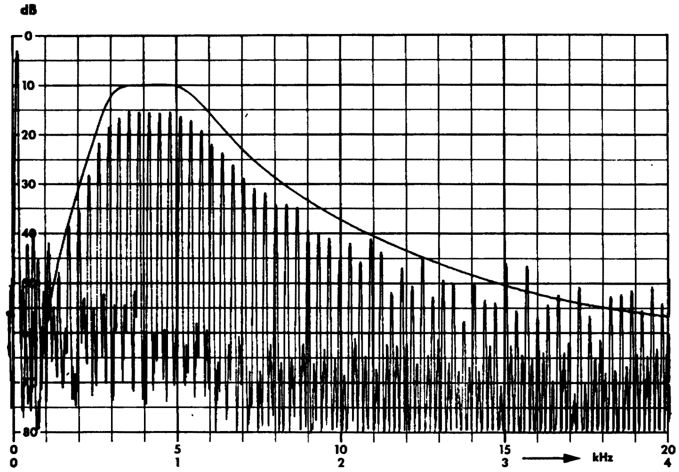

Fig. 5. Spectrum associated with the result shown in Fig. 4, above. The measured correlation function was scanned 64 times/second. Full-drawn line: frequency response of the filter as measured directly (with proper driving and load resistances) by combination of Rohde and Schwarz Audio-Frequency Spectrograph type FNA and the associated Synchronous Oscillator. Full width of spectrum: 4 kHz, linearly divided. Vertical scale in decibels.

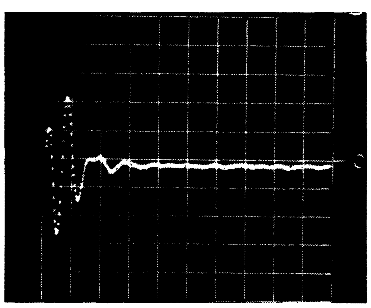

Fig. 6. Result of a computer run. The method is slightly different from that employed in Fig. 4. See text. Note the somewhat larger errors despite the fact that 10 000 sync pulses have been processed as in Fig. 4.

the peaks indicates the Fourier transform of the measured correlation function. This is seen to agree surprisingly well with the frequency response (full drawn line inserted in the figure) of the filter as measured directly. Fig. 6, finally, gives a correlation result obtained in the same situation, but with $x(t)$ triggering the computer and $y(t)$ being fed to the signal input of the computer. In this case, the correlation function appears, as Fig. 6 shows, in the usual form, from left to right. It is seen that in this somewhat artificial case, it does represent the network's impulse response well. The random errors in the result seem larger here, however, since successive y-signal fragments, as processed by the computer, are no longer sections of white noise. That means random errors for different values of τ tend to become correlated.

Application to Neural Excitation Processes

The initiation of nerve impulses can often be described as a triggering process similar to Case 2 treated here. That implies that the theory can be applied to these cases. In fact, the theory was originally formulated to assist in the analysis of inner-ear action. Consider the

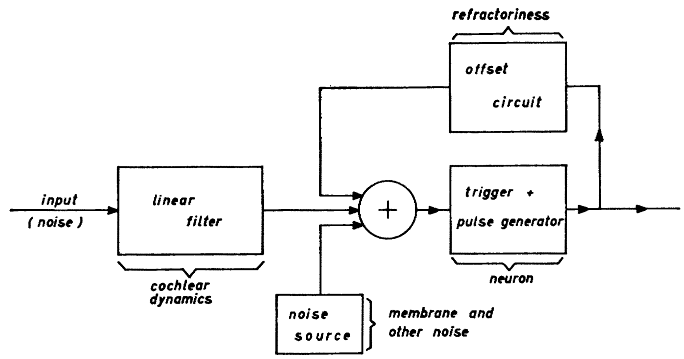

Fig. 7. Weiss' model for the excitation of nerve impulse in the inner ear (simplified).

model of Fig. 7, a model of the excitation of a single nerve fiber in the inner ear (cochlea) by sound. The model has been proposed by Weiss [13] in a study dedicated to simulate the signal transformations in the inner ear. Statistical properties of simulated nerve firings were compared with published histograms of auditory nerve impulses [14]. In the model the linear filter portrays the filtering action of the cochlea at the location of the neuron. Its frequency characteristic shows rather limited frequency selectivity; a band-pass characteristic with a low-frequency slope of 6 dB/octave and a high-frequency slope of 20 dB/octave [13]. The trigger/pulse generator in the model represents the basic stimulus-response relation of a neuron. To account for spontaneous discharges—in the absence of sound—an extra noise source has been included. In addition, this represents the fundamentally random nature of nerve impulse trains. Even in this simple form, the model can account for many observations on nerve impulses in auditory nerve fibers. Some remaining problems required the inclusion of a no-memory nonlinear network interposed before the trigger/pulse generator [13].

The model of Fig. 7 is similar to the procedure of triggered correlation, Case 2. As a matter of fact, Fig. 7, excluding the source of extra noise, could have been used as the block diagram of the system studied in the preceding section. The addition of extra noise, as long as it is independent of the stimulus, does not alter this. Nor does a no-memory nonlinear network interposed before the trigger circuit do any harm. In the case of the auditory system, then, it should be possible to recover the properties of the linear filter by correlation. One should measure the cross-correlation function of the acoustic input signal and the train of nerve impulses at the output. This we can do by synchronizing the computer with the nerve impulses, and letting it process waveform sections of the input noise that occur just prior to these impulses. When white noise is used at the input, the correlogram will be a linear combination of the linear filter's impulse response and its time derivative. In view of the poor cochlear frequency resolution, one would expect a highly damped correlogram. Fig. 8 shows a typical experimental result for an auditory

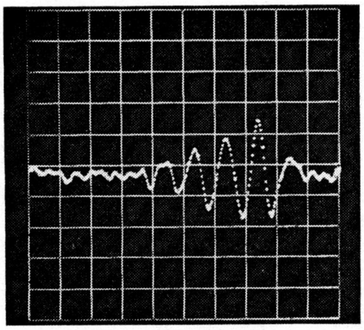

Fig. 8. Correlogram for an auditory neuron (67-12-16). White noise acoustic input. Frequency of greatest sensitivity of the neuron: 1500 Hz. Location of the electrode: auditory nerve (histologically verified). In the processing, the reproduction was at half the original tape speed; the noise was band-limited to 200 to 2000 Hz; the computer was started at the rising edges of the neural spikes. Computer analysis time (full-width screen): 15 ms. Number of spikes processed: 4403.

neuron [15]. It shows, contrary to expectation, that a very material amount of frequency selectivity is involved in the firing of a neuron. Further analysis shows, for instance, that the associated frequency characteristic has a maximum slope of the order of 120 dB/octave.

It was known that with pure-tone stimulation, an auditory neuron is very frequency-selective. The modified method of triggered correlation shows that this selectivity remains demonstrable when the ear is stimulated with white noise. As a matter of fact, the selectivity (as judged by the slope of the associated frequency characteristic) is hardly less than that for tones. If the model applies in this case, the linear-filter part of it must be considerably more frequency-selective than the filtering action of cochlear dynamics. This apparent sharpening, incidentally, serves also to explain the remaining discrepancies between Weiss' original model and experimental data. The next point is the question whether the modified model in its generality does indeed fit the data or not. Discussion in terms of this model actually implies linearization of the processes in the cochlea. Frequency selection then automatically assumes the character of linear filtering, while nonlinearities show up as no-memory nonlinearities. Whether this is legitimate or not remains to be seen. In an attempt to answer this question, efforts are at present directed toward analyzing cochlear action in terms of higher-order nonlinearities. With a technique that is essentially an extension of the one described here, higher-order correlation functions can be measured to provide the experimental basis for such an analysis [16].

Concluding Remarks

Several methods can be devised to let an average-response computer produce an approximation to a cross-correlation function. The simplest methods entail direct production of sync pulses for the computer with help of a trigger circuit. The derivations presented above have shown that such procedures are quite successful. They easily produce a good estimate of the desired correlation function. Overall accuracy is minimally 3.7 dB worse than that for true correlation. This is the penalty we must pay in exchange for elegance, simplicity, and speed. The quality of the obtained correlation function is so good that it allows for further processing, e.g., the measurement of its associated Fourier spectrum. An example of this has been presented.

One of the various methods treated, the simplest one, Case 2, has a larger scope. It indicates a specific way to analyze a compound system consisting of a linear circuit followed by a trigger/pulse generator. It is shown that such compound systems can be analyzed by taking noise as the input signal, and computing the input–output cross-correlation function. The resulting function is closely related to the impulse response of the linear circuit included in the system. This view of triggered correlation may provide a novel method for studying certain neurophysiologic mechanisms. It has already been found useful in the study of the inner ear.

Appendix I

Relay Correlation

Consider the signal values $x=x(t+\tau)$ and $y=y(t)$ as two random variables, x and y, that have a joint Gaussian probability density function. Assume these have unity variance, zero mean, and a correlation coefficient ρ. Then ρ is, in fact, the value of the correlation function $\phi_{xy}(\tau)$ at the pertinent value of τ, and

$$p(x, y) = c \exp \left\{ - \frac{x^2 - 2\rho xy + y^2}{2(1-\rho^2)} \right\}$$

with

$$c = \frac{1}{2\pi(1-\rho^2)^{1/2}} \cdot$$

The relay correlation ρ_{xyr} is calculated by replacing one of the variables, in this case y, by $+1$ when it is positive, and by -1 when it is negative. Price's theorem [17] can be used to analyze such a situation. It states that, when the Gaussian variables x and y are transformed to x' and y', the expectation $\overline{x' \cdot y'}$ of the product $x' \cdot y'$ satisfies

$$\frac{\partial \overline{x'y'}}{\partial \rho} \left(= \frac{\partial E(x'y')}{\partial \rho} \right) = E\left(\frac{dx'}{dx} \cdot \frac{dy'}{dy} \right).$$

In our case $x'=x$ and $y'=\text{sign } (y)$, hence

$$\frac{\partial \overline{x'y'}}{\partial \rho} = c \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} 2\delta(y) \exp \left\{ - \frac{x^2 - 2\rho xy + y^2}{2(1-\rho^2)} \right\} dxdy$$

$$= 2c \int_{-\infty}^{+\infty} \exp \left\{ - \frac{x^2}{2(1-\rho^2)} \right\} dx = \left(\frac{2}{\pi} \right)^{\frac{1}{2}}.$$

For $\rho=0$, $\overline{x'y'}=0$; hence, the integration constant is zero. The final result, valid for Gaussian variables, reads

$$\rho_{xyr} = \overline{x'y'} = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \rho. \tag{11}$$

Note that ρ_{xyr} originally is defined as a covariance, namely as $\overline{x'y'}$. Since our variables x and y are normalized there is no objection against calling ρ_{xyr} a correlation coefficient here.

We now proceed to remove these restrictions. Since Price's theorem holds only for variables with unity variance, we must take the expectation with respect to the original x and y, and use various new substitutions. If we first substitute $x'=\sigma_x x$ and $y'=\text{sign}\ (\sigma_y y)$, we cover the case of centered variables with variances σ_x^2 and σ_y^2, respectively. We then find

$$\overline{x'y'} = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \rho \cdot \sigma_x.$$

To avoid confusion, one should not use the symbol ρ_{xyr} here. If we now substitute $x'=\sigma_x x+\mu_x$, $y'=\sigma_y y+\mu_y$ and $y''=\text{sign}\ y'$, we get

$$\begin{aligned}
\frac{\partial \overline{x'y''}}{\partial \rho} &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \sigma_x \cdot 2\delta(\sigma_y y + \mu_y)\cdot p(x, y)dxdy \\
&= 2c\sigma_x \int_{-\infty}^{+\infty} \exp\left[-\frac{x^2 + (2\rho x\mu_y/\sigma_y) + \mu_y^2/\sigma_y^2}{2(1-\rho^2)}\right]dx \\
&= 2c\sigma_x \int_{-\infty}^{+\infty} \exp\left[-\frac{(x+\rho\mu_y/\sigma_y)^2}{2(1-\rho^2)}\right] \\
&\quad \cdot \exp\left(-\frac{\mu_y^2}{2\sigma_y^2}\right)dx \\
&= \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \sigma_x \exp\left(-\frac{\mu_y^2}{2\sigma_y^2}\right).
\end{aligned}$$

The integration constant is no longer zero. It can be found by direct computation of $\overline{y'x''}$ for the case $\rho=0$. We then find the following two terms, first

$$\begin{aligned}
x^+ &= \int_0^{\infty}\int_{-\infty}^{+\infty} x' p(x', y')dx'dy' \\
&= c_0 \int_{-\mu_y/\sigma_y}^{\infty}\int_{-\infty}^{+\infty} (\sigma_x x + \mu_x) \exp\left(-\frac{x^2+y^2}{2}\right)dxdy
\end{aligned}$$

as the contribution from positive y'-values and, similarly,

$$x^- = c_0 \int_{-\infty}^{-\mu_y/\sigma_y}\int_{-\infty}^{+\infty} (\sigma_x x + \mu_x) \exp\left(-\frac{x^2+y^2}{2}\right)dxdy,$$

where

$$c_0 = (2\pi)^{-1},$$

as the contribution from negative y'-values. Carrying out the integration over x, one finds

$$x^+ = (2\pi)^{-\frac{1}{2}}\mu_x \int_{-\mu_y/\sigma_y}^{+\infty} \exp\left(-\frac{y^2}{2}\right)dy \tag{12}$$

$$x^- = -(2\pi)^{-\frac{1}{2}}\mu_x \int_{-\infty}^{-\mu_y/\sigma_y} \exp\left(-\frac{y^2}{2}\right)dy \tag{13}$$

and by combination

$$[\overline{x'y'}]_{\rho=0} = x^+ - x^- = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\mu_x \cdot \int_0^{+\mu_y/\sigma_y} \exp\left(-\frac{y^2}{2}\right)dy.$$

Hence, the final result reads

$$\begin{aligned}
\overline{x'\cdot y'} = &\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \cdot \mu_x \int_0^{\mu_y/\sigma_y} \exp\left(-\frac{y^2}{2}\right)dy \\
&+ \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \rho\cdot\sigma_x\cdot \exp\left(-\frac{\mu_y^2}{2\sigma_y^2}\right).
\end{aligned} \tag{14}$$

The same method can be applied to polarity correlation [17], [18]. We easily derive for the case of centered variables with unity variance

$$\frac{\partial \rho_{xyp}}{\partial \rho} = \frac{1}{2\pi(1-\rho^2)^{\frac{1}{2}}}.$$

This gives after integration ($\rho_{xyp}=0$ when $\rho=0$)

$$\rho_{xyp} = \left(\frac{2}{\pi}\right) \arcsin \rho. \tag{15}$$

Except for $\rho=1$, the relation between ρ and ρ_{xyp} is monotonic. Zero crossings and maxima, for instance, of the correlation function $\phi_{xy}(\tau)$ occur at the same values of τ in $\phi_{xyp}(\tau)$, hence, the general shape of $\phi_{xy}(\tau)$ is reasonably well retained. When all correlations are small, $\phi_{xyp}(\tau)$ is even proportional to $\phi_{xy}(\tau)$.

Appendix II

Single-Polarity Triggered Correlation

We proceed to derive the average value $\bar{x}_+$ of x under the conditions $y=b$ and $z\geq 0$

$$\bar{x}_+ = \frac{\int xp(x \mid y=b, z\geq 0)dxdydz}{\int p(x \mid y=b, z\geq 0)dxdydz}.$$

Let the correlation matrix $\{R\}$ of x, y and z be

$$\{R\} = \begin{Bmatrix} 1 & \rho_{xy} & \rho_{xz} \\ \rho_{xy} & 1 & 0 \\ \rho_{xz} & 0 & 1 \end{Bmatrix}.$$

The expression for the joint probability density function $p(x, y, z)$ involves elements of the inverse of the matrix $\{R\}$ [19]. The associated characteristic function $C(u_1, u_2, u_3)$ is a relatively simple function:

$$C(u_1, u_2, u_3) = \exp\left\{-\tfrac{1}{2}(u_1^2 + u_2^2 + u_3^2 + 2\rho_{xy}u_1u_2 + 2\rho_{xz}u_1u_3)\right\}.$$

We shall base the calculation on this function. We compute $\bar{x}_+$ in two steps, one for the condition $y=b$, and one to take care of $z \geq 0$. The first step includes computation of the characteristic function $D(u_1, u_3)$ associated with the conditional probability density $q(x, z \mid y=b)$. The latter function can be defined as

$$q(x, z \mid y = b) = k\int_{-\infty}^{+\infty} p(x, y, z)\delta(y - b)dy.$$

The meaning of k is as stated earlier. This relation can be interpreted in several ways. In terms of characteristic functions, the multiplication under the integral sign is equivalent to convolution of $C(u_1, u_2, u_3)$ with the function $k \exp(iu_2b)$. The integration then implies a reduction to the marginal distribution of x and z, conditional on $y=b$. These two procedures are executed as follows. The convolution in the u_2-domain becomes

$$C'(u_1, u_2, u_3) = k\int_{-\infty}^{+\infty} C(u_1, \eta, u_3) \exp ib(u_2 - \eta)d\eta.$$

Grouping of the relevant terms under the integral sign gives rise to the integral

$$I = \int_{-\infty}^{+\infty} \exp\left\{-\tfrac{1}{2}(\eta^2 + 2\rho_{xy}u_1\eta + 2ib\eta\right)d\eta$$
$$= (2\pi)^{\frac{1}{2}} \exp \tfrac{1}{2}(\rho_{xy}u_1 + ib)^2.$$

The complete characteristic function $C'(u_1, u_2, u_3)$ becomes

$$C'(u_1, u_2, u_3) = \exp\left\{-\tfrac{1}{2}(u_1^2 + 2\rho_{xz}u_1u_3 + u_3^2) + \tfrac{1}{2}b^2 + ibu_2 + \tfrac{1}{2}(\rho_{xy}u_1 + ib)^2\right\}.$$

As a function of u_2, it behaves as $\exp ibu_2$, which form corresponds to $\delta(y-b)$ in the y-domain. To obtain the characteristic function $D(u_1, u_3)$, corresponding to the marginal x-z distribution, we merely have to put u_2 equal to zero.

$$D(u_1, u_3) = \exp\left\{-\tfrac{1}{2}(u_1^2 + 2\rho_{xz}u_1u_3 + u_3^2) + \tfrac{1}{2}b^2 + \tfrac{1}{2}(\rho_{xy}u_1 + ib)^2\right\}. \quad (16)$$

This completes the first step. The result is seen to be the characteristic function of two jointly distributed Gaussian variables.

$$D(u_1, u_3) = \exp\left\{-\tfrac{1}{2}(u_1^2\sigma_1^2 + 2\rho u_1u_3\sigma_1\sigma_3 + u_3^2\sigma_3^2) + i(\mu_1u_1 + \mu_3u_3)\right\}$$

where σ_1^2 and σ_3^2 are the variances, and μ_1 and μ_3 the means of these variables, and ρ is the correlation coefficient. Thus we obtain the following identifications:

$$\sigma_1^2 = 1 - \rho_{xy}^2$$
$$\sigma_3^2 = 1$$
$$\rho\sigma_1\sigma_3 = \rho_{xz}$$
$$\mu_1 = b\rho_{xy}$$
$$\mu_3 = 0.$$

As the second step we must compute the average value $\bar{x}_+$ of x over the obtained distribution, but averaged under the condition $z \geq 0$. Relation (12) gives the contribution x^+ under the condition $z \geq 0$. Since $\mu_3 = 0$, the condition occurs with probability $\frac{1}{2}$. Hence, twice this term is the desired $\bar{x}_+$. With the remaining identifications one obtains directly, when adding the second term of (14)

$$\bar{x}_+ = b\rho_{xy} + \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\rho_{xz}. \quad (17)$$

References

[1] J. S. Bendat, *Principles and Applications of Random Noise Theory.* New York: Wiley, 1958, p. 70.
S. Goldman, *Information Theory.* New York: Prentice-Hall, 1955, p. 278.
W. B. Davenport and W. L. Root, *Random Signals and Noise.* New York: McGraw-Hill, 1958, p. 182.
H. Schlitt, *Systemtheorie für regellose Vorgänge.* Berlin: Springer, 1960, p. 201.

[2] A. H. Nuttal, "Invariance of correlation functions under nonlinear transformations," M.I.T. Research Lab. of Electronics, Cambridge, Mass., Quart. Prog. Rept., p. 69, April 1957.

[3] Y. W. Lee, "Application of statistical methods to communication problems," M.I.T. Research Lab. of Electronics, Cambridge, Mass., Tech. Rept. 181, 1950.
Y. W. Lee, T. P. Cheatham, Jr., and J. B. Wiesner, "Application of correlation analysis to the detection of periodic signals in noise," *Proc. IRE*, vol. 38, pp. 1165–1171, October 1950.

[4] M.I.T. Communication Biophysics Group and W. M. Siebert, "Processing neuroelectric data," M.I.T. Research Lab. of Electronics, Cambridge, Mass., Tech. Rept. 351, 1959.

[5] B. P. Th. Veltman and A. van den Bos, "The applicability of the relay-correlator and the polarity-coincidence correlator in automatic control," *Proc. Internat'l Federation of Automatic Control* (Basel, Switzerland), 1963.
D. G. Watts, "A general theory of amplitude quantization with applications to correlation determination," IEE (London) Monograph 481 M, November 1961.

[6] B. P. Th. Veltman and H. Kwakernaak, "Theorie und Technik der Polaritätskorrelation usw.," *Regelungstechn*, vol. 9, pp. 357–364, 1961.

[7] Y. W. Lee, *Statistical Theory of Communication.* New York: Wiley, 1965, p. 303.

[8] A. Kamp, W. S. van Leeuwen, and A. M. Thielen, "A method for auto- and cross-relation analysis of the E.E.G.," *Electroencephal. and Clinical Neurophysiol.*, vol. 19, pp. 91–95, 1965.

[9] W. B. Davenport and W. L. Root, *op. cit.*, p. 93.
S. O. Rice, "Mathematical analysis of random noise," *Bell Sys. Tech. J.*, vol. 23, pp. 282–332, 1944, and vol. 24, pp. 46–156, 1945, also in N. Wax, *Noise and Stochastic Processes.* New York: Dover, 1954, p. 133–294, see especially sec. 1.7.
J. S. Bendat, *op. cit.*, pp. 80, 138.

[10] J. Ville, "Theorie et application de la notion de signal analytique," *Cables et Transmission*, pp. 61–74, 1948.

J. R. V. Oswald, "The theory of analytic band-limited signals applied to carrier systems," *IRE Trans. Circuit Theory*, vol. CT-3, pp. 244–251, December 1956.
R. Deutsch, *Nonlinear Transformations of Random Processes*. Englewood Cliffs, N. J.: Prentice-Hall, 1962, ch. 1.
[11] S. O. Rice, *op. cit.*, sec. 3.3.
J. S. Bendat, *op. cit.*, ch. 3.4, pp. 124–133.
[12] E. de Boer, "De nauwkeurigheid van correlatie methoden" (Accuracy of correlation methods), *Ned. Tijdschr. Natuurk.*, vol. 30, pp. 275–287, 1964.
[13] Th. Weiss, "A model for firing patterns of auditory nerve fibers," M.I.T. Research Lab. of Electronics, Cambridge, Mass., Tech. Rept. 418, 1964; and "A model of the peripheral auditory system," *Kybernetik.*, vol. 3, pp. 153–175, 1966.
[14] N. Y. S. Kiang, T. Watanabe, E. C. Thomas, and L. F. Clark' *Discharge Patterns of Single Fibers in the Cat's Auditory Nerve*· Cambridge, Mass.: M.I.T. Press, 1965.
[15] E. de Boer, "Correlation studies applied to the frequency resolution of the cochlea," *J. Auditory Research*, vol. 7, pp. 209–217, 1967.
[16] ——, "Higher-order triggered correlation" (to be published).
[17] R. Price, "A useful theorem for nonlinear devices having Gaussian inputs," *IRE Trans. Information Theory*, vol. IT-4, pp. 69–72, June 1958.
[18] J. H. van Vleck and D. Middleton, "The spectrum of clipped noise," *Proc. IEEE*, vol. 54, pp. 2–19, January, 1966.
[19] H. Cramér, *Mathematical Methods of Statistics*. Princeton N. J.: Princeton University Press, 1958, p. 118 ff and p. 310 ff

The Solution of Overdetermined Linear Equations as a Multistage Process

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***Abstract*—The concepts of dynamic programming and multistage process theory are used to determine an analytic, least-squares solution to overdetermined linear algebraic equations. The method requires the calculation of a secondary system matrix which, together with the primary matrix, makes evident the relative dependence of the system equations. It is shown that the minimum-sum-square residual, a measure of the fit of the solution, can be determined without explicit calculation of the solution.**

I. Introduction

IN THE STUDY of systems, especially those of biological origin, one is often faced with the necessity of solving large sets of overdetermined, incompatible, linear algebraic equations.[1] Such equations can be solved only by applying some constraint. A commonly used constraint is the method of least squares which requires that the sum of the squared residuals be a minimum.

This paper presents a method of solution that is developed using the concepts of dynamic programming and multistage process theory [2]. The theory is employed, in concept only, to derive a set of analytic recurrence equations with which the unknowns can be calculated. The minimum sum of the squared residuals can be expressed in closed form, independent of the unknowns of the system. A secondary system matrix is calculated in the process of the solution, and this secondary matrix, together with the primary matrix, makes evident the relative dependence of the system equations.

The results of this method of solution are summarized as follows. The solution of the set

$$\Phi - \sum_{j=1}^{N} C_j p_j = R \tag{1}$$

subject to the constraint

$$f(\Phi) = \min_{\text{all } p} (R^2), \tag{2}$$

where Φ, C_j, and R are M-dimensional vectors with $M \geq N$, can be determined as

$$p_k = \frac{E_k}{E_k^2} \cdot \left(\Phi - \sum_{j=k+1}^{N} C_j p_j \right), \tag{3}$$

where

$$E_k = C_k - \sum_{r=1}^{k-1} E_r \frac{E_r \cdot C_k}{E_r^2} \cdot \tag{4}$$

The minimum-sum-squared residual is given by

$$f(\Phi) = \Phi \cdot \left(\Phi - \sum_{k=1}^{N} E_k \frac{E_k \cdot \Phi}{E_k^2} \right). \tag{5}$$

Manuscript received May 12, 1967; revised December 28, 1967. The research presented here was supported in part by the National Institutes of Health, under Grant HE 05716-01.
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[1] An overdetermined set is one having more equations than unknowns. An incompatible set is one which no set of unknowns can identically satisfy. See Lanczos [1].